



# A novel route to $V_{us}$

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Already in the past, hadronic  $\tau$  decays have served as an interesting source to obtain information on the parameters of the Standard Model, like the strong coupling  $\alpha_s$  or the strange quark mass. Below it will be shown that the approach to obtain the strange mass from the hadronic  $\tau$ -decay width can be turned into a determination of the CKM matrix element  $V_{us}$ , once the strange mass is used as an input from other sources. At present, we obtain  $V_{us} = 0.2179 \pm 0.0045$ , where the error is completely dominated by the experimental uncertainty, and can be improved through an improved measurement of the hadronic  $\tau$ -decay rate into strange particles  $R_{\tau,S}$ .

## 1 Introduction

Already more than a decade ago it was realised that the hadronic decay of the  $\tau$  lepton could serve as an ideal system to study low-energy QCD under rather clean conditions [1]. In the following years, detailed investigations of the  $\tau$  hadronic width as well as invariant mass distributions have served to determine the QCD coupling  $\alpha_s$  to a precision competitive with the current world average [2,3]. More recently, the experimental separation of the Cabibbo-allowed decays and Cabibbo-suppressed modes into strange particles opened a means to also determine the mass of the strange quark [4–12], one of the fundamental QCD parameters within the Standard Model.

Until recently, strange quark mass determinations from hadronic  $\tau$  decays suffered from sizeable uncertainties due to higher order perturbative corrections. These result from large QCD corrections to the contributions of scalar and pseudoscalar correlation functions [1,12–14] which are additionally amplified by the particular weight functions which appear in the  $\tau$  sum rule. However, a natural remedy to circumvent this problem is to replace the QCD expressions of scalar and pseudoscalar correlators by corresponding phenomenological hadronic parametrisations [4,15,7,9,10], which turn out to be more precise than their QCD counterparts, since the by far dominant contribution stems from the known kaon pole.

Additional suppressed contributions to the pseudoscalar correlators come from the pion pole as well as higher excited pseudoscalar states whose parameters have recently been estimated [16]. The remaining strangeness-changing scalar spectral function has been extracted very recently from a study of S-wave  $K\pi$  scattering [17,18] in the framework of chiral perturbation theory ( $\chi$ PT) with explicit inclusion of resonances. The resulting scalar spectral function was then employed to directly determine the strange quark mass from a purely scalar QCD sum rule [19]. On the other hand, now we are also in a position to incorporate this contribution into the  $\tau$  sum rule.

Nevertheless, as was already realised in the first works on the strange mass determination from the Cabibbo-suppressed  $\tau$  decays,  $m_s$  turns out to depend sensitively on the element  $V_{us}$  of the quark-mixing (CKM) matrix. With the theoretical improvements in the  $\tau$  sum rule mentioned above, in fact  $V_{us}$  represents one of the dominant uncertainties in the determination of the strange mass. Thus it appears natural to turn things around and, with an input of  $m_s$  as obtained from other sources, to actually determine  $V_{us}$ . It is then found that by far the dominant error on  $V_{us}$  results from the experimental uncertainty on the hadronic  $\tau$ -decay rate into strange particles  $R_{\tau,S}$ , which should be improvable in future experimental analyses.

## 2 Theoretical framework

The main quantity of interest for the following analysis is the hadronic decay rate of the  $\tau$  lepton,

$$R_\tau \equiv \frac{\Gamma(\tau^- \rightarrow \text{hadrons } \nu_\tau(\gamma))}{\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma))} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}. \quad (1)$$

Theoretically, it can be expressed as an integral of the spectral functions  $\text{Im} \Pi^T(s)$  and  $\text{Im} \Pi^L(s)$  over the invariant mass  $s = p^2$  of the final state hadrons [1],

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[ \left(1 + \frac{2s}{M_\tau^2}\right) \text{Im} \Pi^T(s) + \text{Im} \Pi^L(s) \right]. \quad (2)$$

The appropriate combinations of two-point correlation functions which appear in eq. (2) are given by

$$\begin{aligned} \Pi^J(s) \equiv & |V_{ud}|^2 \left[ \Pi_{ud}^{V,J}(s) + \Pi_{ud}^{A,J}(s) \right] \\ & + |V_{us}|^2 \left[ \Pi_{us}^{V,J}(s) + \Pi_{us}^{A,J}(s) \right], \end{aligned} \quad (3)$$

with  $V_{ij}$  being the corresponding matrix elements of the CKM matrix. As has been indicated in eq. (1), experimentally, one can disentangle vector from axialvector contributions in the Cabibbo-allowed ( $\bar{u}d$ ) sector, whereas such

a separation is problematic in the Cabibbo-suppressed ( $\bar{u}s$ ) sector. The superscripts in the transversal and longitudinal components denote the corresponding angular momentum  $J = 1$  ( $T$ ) and  $J = 0$  ( $L$ ) in the hadronic rest frame.<sup>1</sup>

Additional information can be inferred from the measured invariant mass distribution of the final state hadrons. The corresponding moments  $R_\tau^{kl}$ , defined by [20]

$$R_\tau^{kl} \equiv \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds} = R_{\tau,V+A}^{kl} + R_{\tau,S}^{kl}, \quad (4)$$

can be calculated theoretically in analogy to  $R_\tau = R_\tau^{00}$ . In the framework of the operator product expansion (OPE),  $R_\tau^{kl}$  can be written as [1]:

$$R_\tau^{kl} = 3 S_{\text{EW}} \left\{ \left( |V_{ud}|^2 + |V_{us}|^2 \right) \left( 1 + \delta^{kl(0)} \right) + \sum_{D \geq 2} \left( |V_{ud}|^2 \delta_{ud}^{kl(D)} + |V_{us}|^2 \delta_{us}^{kl(D)} \right) \right\}. \quad (5)$$

The electroweak radiative correction  $S_{\text{EW}} = 1.0201 \pm 0.0003$  [21–23] has been pulled out explicitly, and  $\delta^{kl(0)}$  denotes the purely perturbative dimension-zero contribution. The symbols  $\delta_{ij}^{kl(D)}$  stand for higher dimensional corrections in the OPE from dimension  $D \geq 2$  operators which contain implicit suppression factors  $1/M_\tau^D$ .

The separate measurement of Cabibbo-allowed as well as Cabibbo-suppressed decay widths of the  $\tau$  lepton [10] allows one to pin down the flavour SU(3)-breaking effects, dominantly induced by the strange quark mass. Defining the difference

$$\delta R_\tau^{kl} \equiv \frac{R_{\tau,V+A}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2} = 3 S_{\text{EW}} \sum_{D \geq 2} \left( \delta_{ud}^{kl(D)} - \delta_{us}^{kl(D)} \right), \quad (6)$$

many theoretical uncertainties drop out since these observables vanish in the SU(3) limit. In particular, they are free of possible flavour-independent instanton as well as renormalon contributions which could mimic dimension-two corrections.

### 3 Longitudinal contributions

Before we discuss the determination of  $m_s$  and  $V_{us}$  from hadronic  $\tau$  decays, let us first comment on the longitudinal contributions to eq. (2). As was already remarked above, these contributions are plagued with huge perturbative higher order corrections, which in previous analyses resulted in large corresponding uncertainties for the strange

|                   | Phenom:                          | Theory:                          |
|-------------------|----------------------------------|----------------------------------|
| $R_{us,A}^{00,L}$ | $-0.135 \pm 0.003$               | $-0.144 \pm 0.024$               |
| $R_{us,V}^{00,L}$ | $-0.028 \pm 0.004$               | $-0.028 \pm 0.021$               |
| $R_{ud,A}^{00,L}$ | $-(7.77 \pm 0.08) \cdot 10^{-3}$ | $-(7.79 \pm 0.14) \cdot 10^{-3}$ |

**Table 1.** Comparison of theoretical and phenomenological longitudinal contributions to the (0,0) moment of the  $\tau$  sum rule.

quark mass. This problem can be circumvented by replacing the theoretical expressions with corresponding contributions resulting from phenomenological parametrisations of the relevant spectral functions.

A numerical comparison of theoretical and phenomenological longitudinal contributions  $R_{ij,V/A}^{kl,L}$  for the (0,0) moment is presented in table 1.<sup>2</sup> On the phenomenological side, the dominant axialvector ( $us$ ) contribution originates from the strange pseudoscalar mesons, by far the largest part coming from the kaon pole. Also the next two higher excited resonances have been included with a standard Breit-Wigner resonance shape and resonance parameters as recently estimated in [16]. Similarly, the axialvector ( $ud$ ) contribution is due to the pion pole as well as higher excited states. However, because of the smaller pion mass, its contribution is much suppressed. The dominant uncertainties in the phenomenological results in table 1 are due to the errors in the decay constants  $f_K$  and  $f_\pi$ , as well as the higher resonance decay constants.

The longitudinal vector ( $us$ ) contribution is related to the corresponding scalar spectral function which has been calculated in ref. [19]. Below 2 GeV the dominant hadronic systems which contribute in this channel are the  $K\pi$  and  $K\eta'$  states with  $K_0^*(1430)$  being the lowest lying scalar resonance. The scalar  $us$  spectral function can then be parametrised in terms of the scalar  $F_{K\pi}(s)$  and  $F_{K\eta'}(s)$  form factors. These form factors were obtained in [18] from a coupled-channel dispersion-relation analysis. The S-wave  $K\pi$  scattering amplitudes which are required in the dispersion relations were available from a description of S-wave  $K\pi$  scattering data in the framework of unitarised  $\chi$ PT with resonances [17].

Generally, from table 1 one observes that phenomenological and theoretical values for the  $R_{ij,V/A}^{00,L}$  are in rather good agreement. Nevertheless, because of the large uncertainties from higher order perturbative corrections in the theoretical expressions, the phenomenological values are more precise and thus in the following they will be employed for the analysis of the hadronic  $\tau$  decays.

<sup>1</sup>Further details on our notation can be found in refs. [1,4].

<sup>2</sup>For the precise definition of the  $R_{ij,V/A}^{kl,L}$  see ref. [4]

## 4 Strange quark mass

The leading term in the SU(3)-breaking difference of eq. (6) is due to the dimension-2 contribution proportional to  $m_s^2$ . Thus it appears natural to determine the strange quark mass by comparing experimental and theoretical results for  $\delta R_\tau^{kl}$ . In practice, the moments (0,0) to (4,0) have been utilised in the phenomenological analysis. For low  $k$ , the higher-energy region of the experimental spectrum, which is less well known, plays a larger role and thus in this region the experimental uncertainties dominate the strange mass determination, whereas for higher  $k$  more emphasis is put on the lower-energy region, and there the theoretical uncertainties dominate.

In addition, as will be exploited in more detail in the next section, the strange mass extracted from the  $\tau$  sum rules also displays a sizeable dependence on  $V_{us}$ , which is strongest for  $k = 0$  and becomes weaker for larger  $k$ . The smallest combined uncertainty is then found for the (3,0) moment which with present experimental and theoretical errors represents an optimal choice. Taking a weighted average of the strange mass values obtained for the different moments we then find

$$m_s(2 \text{ GeV}) = 103 \pm 17 \text{ MeV}, \quad (7)$$

where the uncertainty corresponds to that of the (3,0) moment, and the value employed for  $V_{us}$  is the central PDG average  $|V_{us}| = 0.2196 \pm 0.0026$  [24], based on the analyses [25–27]. (See also ref. [28].) To illustrate the dependence of  $m_s$  on  $V_{us}$ , one can compare to the result when using the value  $|V_{us}| = 0.2225 \pm 0.0021$  obtained from a fit imposing unitarity on the CKM matrix [24]. The strange quark mass is then found to be

$$m_s(2 \text{ GeV}) = 117 \pm 17 \text{ MeV}. \quad (8)$$

The dominant theoretical uncertainties in the results of eqs. (7) and (8) originate from higher order perturbative corrections as well as the SU(3) breaking ratio of the quark condensates  $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle$  [29] which arises in the next order dimension-4 contribution to eq. (6). A detailed discussion of all input parameters and a breakup of the various uncertainties can be found in ref. [4].

Before coming to the determination of  $V_{us}$  from the  $\tau$  sum rule, a somewhat disturbing observation in the strange mass analysis still needs to be discussed. Namely, it is found that  $m_s$  obtained from the lowest (0,0) moment is largest, and then the strange mass continually decreases for the higher moments. Although the individual  $m_s$  values are all within one sigma of the central result, it would be important to find out if this  $k$ -dependence stems from a deficiency in the theoretical description or in the experimental data.

One possible explanation of the effect could be missing contributions in the higher-energy region of the experimental strange spectrum, because these would suppress the strange mass determined from lower- $k$  moments. A hint in this direction is given by a recent CLEO result [30], confirming previous CLEO and OPAL results, which found the branching fraction of the exclusive  $K\pi\pi$  mode to be  $B(\tau^- \rightarrow K^- \pi^+ \pi^- \nu_\tau) = (3.84 \pm 0.14 \pm 0.38) \cdot 10^{-3}$ , nearly 3 sigma higher than the corresponding findings by ALEPH [10], on which the spectral moment analysis is based.

## 5 A novel route to $V_{us}$

In order to calculate  $V_{us}$  from the SU(3)-breaking difference (6), we now require a value for the strange mass from other sources as an input so that we are in a position to calculate  $\delta R_\tau^{kl}$  from theory. In the following, we shall use the result  $m_s(2 \text{ GeV}) = 105 \pm 20 \text{ MeV}$ , a value compatible with most recent determinations of  $m_s$  from QCD sum rules [4,5,19,16,31] and lattice QCD [32]. A compilation of recent strange mass determinations is also displayed in figure 1.

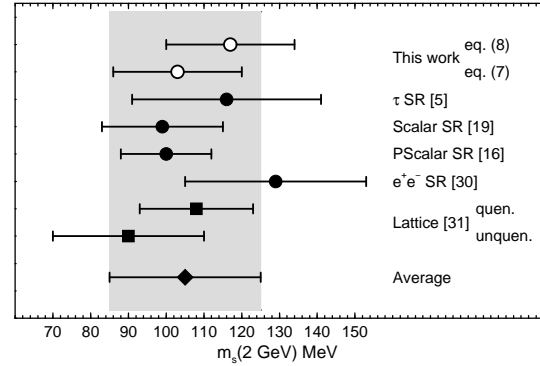


Figure 1. Summary of recent results for  $m_s(2 \text{ GeV})$ .

Since the sensitivity to  $V_{us}$  is strongest for the (0,0) moment, where also the theoretical uncertainties are smallest, this moment will be used for the determination of  $V_{us}$ . Inserting the above strange mass value into the theoretical expression for  $\delta R_{\tau,th}$ , one finds [4]

$$\delta R_{\tau,th} = 0.229 \pm 0.030, \quad (9)$$

where the uncertainty dominantly results from a variation of the strange quark mass within its errors. Assuming unitarity of the CKM matrix in order to express the CKM matrix element  $V_{ud}$  in terms of  $V_{us}$  in eq. (6), we then obtain

$$V_{us} = 0.2179 \pm 0.0044_{\text{exp}} \pm 0.0009_{\text{th}} = 0.2179 \pm 0.0045. \quad (10)$$

The first given error is the experimental uncertainty due to  $R_\tau = 3.642 \pm 0.012$  and, most importantly, the hadronic

$\tau$  decay rate with net strangeness  $R_{\tau,S} = 0.1625 \pm 0.0066$  [33], whereas the second error stems from the theoretical value for  $\delta R_\tau$ . Even though the theoretical error on  $\delta R_{\tau,th}$  is roughly 15%, since individually both terms on the rhs of (6) are much larger, for the extraction of  $V_{us}$ ,  $\delta R_{\tau,th}$  is only a correction and its error rather unimportant. The theoretical uncertainty in  $\delta R_{\tau,th}$  will only start to matter once the experimental error on  $R_{\tau,S}$  is much improved, possibly through an analysis of the BABAR and BELLE  $\tau$  data samples.

## 6 Conclusions

Taking advantage of the strong sensitivity of the flavour-breaking  $\tau$  sum rule on the CKM matrix element  $V_{us}$ , it is possible to determine it from hadronic  $\tau$  decay data. This requires a value of the strange quark mass as an input which can be obtained from other sources like QCD sum rules or the lattice. The result for  $V_{us}$  thus obtained is

$$V_{us} = 0.2179 \pm 0.0045, \quad (11)$$

where the error is largely dominated by the experimental uncertainty on  $R_{\tau,S}$ . A reduction of this uncertainty by a factor of two would result in a corresponding reduction of the error on  $V_{us}$ , which would lead to a determination more precise than the current PDG average.

Furthermore, precise experimental measurements of  $R_{\tau,S}^{kl}$  and the SU(3)-breaking differences  $\delta R_\tau^{kl}$  would open the possibility to determine both  $m_s$  and  $V_{us}$  simultaneously. This can hopefully be achieved with the BABAR and BELLE  $\tau$  data samples in the near future. This is particularly important since a very recent new measurement of  $K_{e3}$  decays [34] points to a larger value of  $V_{us}$ , almost 3 sigma away from the present PDG average.

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## References

1. E. Braaten, S. Narison and A. Pich, *Nucl. Phys. B* **373**, 581 (1992).
2. ALEPH coll. (R. Barate et al.), *Eur. Phys. J. C* **4**, 409 (1998).
3. OPAL coll. (K. Ackerstaff et al.), *Eur. Phys. J. C* **7**, 571 (1999) [hep-ex/9808019].
4. E. Gámiz, M. Jamin, A. Pich, J. Prades and F. Schwab, *JHEP* **01**, 060 (2003) [hep-ph/0212230].
5. S.M. Chen et al., *Eur. Phys. J. C* **22**, 31 (2001) [hep-ph/0105253].
6. M. Davier, S.M. Chen, A. Höcker, J. Prades and A. Pich, *Nucl. Phys. Proc. Suppl.* **98**, 319 (2001).
7. J. Kambor and K. Maltman, *Phys. Rev. D* **62**, 093023 (2000) [hep-ph/0005156].
8. J.G. Körner, F. Krajewski and A.A. Pivovarov, *Eur. Phys. J. C* **20**, 259 (2001) [hep-ph/0003165].
9. A. Pich and J. Prades, *JHEP* **10**, 004 (1999) [hep-ph/9909244].
10. ALEPH coll. (R. Barate et al.), *Eur. Phys. J. C* **11**, 599 (1999) [hep-ex/9903015].
11. K.G. Chetyrkin, J.H. Kühn and A.A. Pivovarov, *Nucl. Phys. B* **533**, 473 (1998) [hep-ph/9805335].
12. A. Pich and J. Prades, *JHEP* **06**, 013 (1998) [hep-ph/9804462].
13. K.G. Chetyrkin and A. Kwiatkowski, *Z. Phys. C* **59**, 525 (1993) [hep-ph/9805232].
14. K. Maltman, *Phys. Rev. D* **58**, 093015 (1998) [hep-ph/9804298].
15. K. Maltman and J. Kambor, *Phys. Rev. D* **64**, 093014 (2001) [hep-ph/0107187].
16. K. Maltman and J. Kambor, *Phys. Rev. D* **65**, 074013 (2002) [hep-ph/0108227].
17. M. Jamin, J.A. Oller and A. Pich, *Nucl. Phys. B* **587**, 331 (2000) [hep-ph/0006045].
18. M. Jamin, J.A. Oller and A. Pich, *Nucl. Phys. B* **622**, 279 (2002) [hep-ph/0110193].
19. M. Jamin, J.A. Oller and A. Pich, *Eur. Phys. J. C* **24**, 237 (2002) [hep-ph/0110194].
20. F. Le Diberder and A. Pich, *Phys. Lett. B* **289**, 165 (1992).
21. W. Marciano and A. Sirlin, *Phys. Rev. Lett.* **61**, 1815 (1988).
22. E. Braaten and C.S. Li, *Phys. Rev. D* **42**, 3888 (1990).
23. J. Erler, [hep-ph/0211345].
24. Particle Data Group (K. Hagiwara et al.), *Phys. Rev. D* **66**, 010001 (2002).
25. H. Leutwyler and M. Roos, *Z. Phys. C* **25**, 91 (1984).
26. V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, *Eur. Phys. J. C* **23**, 121 (2002) [hep-ph/0110153].
27. G. Calderon and G. Lopez Castro, *Phys. Rev. D* **65**, 073032 (2002) [hep-ph/0111272].
28. M. Battaglia, A.J. Buras, P. Gambino and A. Stocchi, eds. *Proceedings of the First Workshop on the CKM Unitarity Triangle*, CERN, Feb 2002, hep-ph/0304132.
29. M. Jamin, *Phys. Lett. B* **538**, 71 (2002) [hep-ph/0201174].
30. CLEO coll. (R.A. Briere et al.), *Phys. Rev. Lett.* **90**, 181802 (2003) [hep-ex/0302028].
31. S. Narison, *Phys. Lett. B* **466**, 345 (1999) [hep-ph/9905264].
32. H. Wittig, [hep-lat/0210025].
33. M. Davier and C.-Z. Yuan, [hep-ex/0209091].
34. BNL-E865 coll. (A. Sher et al.), [hep-ex/0305042]